Explicit models for bilateral fat-tailed distributions
and applications in finance with the package FatTailsR

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About InModelia

A small company located in Paris:
- Consulting services and training
- Modeling and software design

Since 2009:
- Neural networks
- Design of experiments for nonlinear models

Since 2013:
- Bilateral fat-tailed distributions
  (from a work done with ST-Microelectronics)

June 2014:
- Package **FatTailsR** to handle fat-tailed distributions in finance
  Models are symmetric (3 parameters) or asymmetric (4 parameters)
  Excellent results !!

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Outline

1. Old models - New ideas

2. Symmetric models - Examples: Gold, Société Générale, Vivendi

3. Asymmetric models - Examples: S&P 500, Euro-Dollar, VIX

4. Package FatTailsR
Fat-tailed bilateral distributions

- Almost all datasets related to stock returns over long periods
- Market risk (Solvency II, Bâle III), portfolio management, derivatives, ...
- Mentioned by Mandelbrot (1962), reviewed by Bouchaud and Potters (1997), who use a combination of unilateral models for left and right tails

\[ L_\mu(x) \approx \frac{\mu A_\mu}{|x|^{1+\mu}} \text{ pour } x \to \pm \infty, \]

Figure: (a+b) Cotton price day-week-month - (c+d) Indice S&P 15min-day-week-month

- Centered and non-centered Student distributions
- Characteristic functions

\[ \log \{ \text{Fr} [L(t, T) > u] \} \sim -z \log u + \log C'(T). \]
\[ \log \{ \text{Fr} [L(t, T) < -u] \} \sim -z \log u + \log C''(T). \]

\[ \log \{ \text{Fr} [L(t, T) > u] \} \sim -z \log u \]

We present a new, explicit, symmetric or asymmetric distribution
Example: S&P 500

We present a new, explicit, symmetric or asymmetric distribution

Figure: S&P 500: (a) Cumulative function of the logreturns (b) Log-Log view day-week-month
Consider the logistic function which have thinner tails than the Laplace-Gauss function:

\[ F(x) = \frac{1}{1 + e^{-x}} \]

It is remarkable that the combination of two asymmetric functions \( e^{-x} \) et \( \frac{1}{1 + ...} \) gives a perfectly symmetric model. This comes from the fundamental property of the exponential \( e^{-x}e^x = 1 \) which imposes \( F(-x) = 1 - F(x) \) and for the density function \( f(x) = F(x)F(-x) \)

Figure: (a) Exp and hp  (b) logis and logishp  (c) dlogis and dlogishp
New idea

Consider the logistic function which have thinner tails than the Laplace-Gauss function:

\[ F(x) = \frac{1}{1 + e^{-x}} \]

It is remarkable that the combination of two asymmetric functions \( e^{-x} \) et \( \frac{1}{1 + \ldots} \) gives a perfectly symmetric model. This comes from the fundamental property of the exponential \( e^{-x}e^x = 1 \) which imposes \( F(-x) = 1 - F(x) \) and for the density function \( f(x) = F(x)F(-x) \).

![Graphs showing different functions](image)

**Figure:** (a) Exp and hp  (b) logis and logishp  (c) dlogis and dlogishp

We use the other curves that verify the property \( y(-x)y(x) = 1 \):

- **hyperbolas**
- **Power hyperbolas** with parameter \( \kappa \) that allow the design of logistic type functions with tail convergence similar to \( |x|^{-\kappa} \)
Power hyperbolas

We call *power hyperbolas* the positive functions that verify:

### Generic equation of power hyperbolas

\[
\left( y^{1/\kappa} + \frac{X - \mu}{\gamma \kappa} \right) y^{1/\kappa} = 1
\]  

(1)

Solution to the equation are the curves:

\[
y(X, \mu, \gamma, \kappa) = \left( -\frac{X - \mu}{2\gamma \kappa} + \sqrt{\left( \frac{X - \mu}{2\gamma \kappa} \right)^2 + 1} \right)^\kappa = e^{\kappa \log \left( -\frac{X - \mu}{2\gamma \kappa} + \sqrt{\left( \frac{X - \mu}{2\gamma \kappa} \right)^2 + 1} \right)}
\]

or:

### Power hyperbolas

\[
y(X, \mu, \gamma, \kappa) = e^{-\kappa \text{ asinh} \left( \frac{X - \mu}{2\gamma \kappa} \right)} \quad \kappa \to +\infty \quad \Rightarrow \quad e^{-\frac{X - \mu}{2\gamma}}
\]  

(2)

When \( \kappa = 1 \), the curve is the simple hyperbola

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Figure: (a) Power hyperbola - exphp $\kappa(-x) = e^{-\kappa \text{asinh}(\frac{x}{2\kappa})}$ \(\kappa \to +\infty\) \(\to e^{-\frac{x}{2}}\)

(b) Power hyperbolic cosine - coshp $\kappa(x) = \cosh\left[\kappa \text{asinh}(\frac{x}{2\kappa})\right]$ \(\kappa \to +\infty\) \(\to \cosh(\frac{x}{2})\)

(c) Power hyperbolic sine - sinhp $\kappa(x) = \sinh\left[\kappa \text{asinh}(\frac{x}{2\kappa})\right]$ \(\kappa \to +\infty\) \(\to \sinh(\frac{x}{2})\)

(d) Power hyperbolic tangent - tanhp $\kappa(x) = \tanh\left[\kappa \text{asinh}(\frac{x}{2\kappa})\right]$ \(\kappa \to +\infty\) \(\to \tanh(\frac{x}{2})\)
Power hyperbolas, cumulative functions, densities K1

Figure: (a) Power hyperbolas - (b) Cumulative functions - (c) Densities

K1 model: cumulative functions and densities, $T = \text{asinh} \left( \frac{X - \mu}{2\gamma \kappa} \right)$:

\[
F(X) = \frac{1}{1 + e^{-\kappa \text{asinh} \left( \frac{X - \mu}{2\gamma \kappa} \right)}} \\
f(X) = \frac{1}{4\gamma \cosh(T) \left( 1 + \cosh(\kappa T) \right)} \\
f(0) = \frac{1}{8\gamma} \tag{3}
\]

$F$ and $f$ verify Karamata theorem related to slowly varying functions:

\[
\lim_{x \to -\infty} \frac{x f(x)}{F(x)} = -\kappa \quad \text{et} \quad \lim_{x \to +\infty} \frac{x f(x)}{1 - F(x)} = \kappa
\]

Quantiles and densities (calculated from the probability):

\[
X = \mu + 2\gamma \kappa \sinh \left( \frac{\text{logit}(p)}{\kappa} \right) \\
f(X) = \frac{1}{2\gamma} \sech \left( \frac{\text{logit}(p)}{\kappa} \right) p \left( 1 - p \right)
\]
Logit and Logdensities

Figure: (a) QL-Plot = Quantiles + Logit of cumulative functions  (b) Logdensities

\[
\logit F(X, \mu, \gamma, \kappa) = \kappa \ \text{asinh} \left( \frac{X - \mu}{2\gamma\kappa} \right)
\]

\[
\log f(X) = -\log(8\gamma) - \log(\cosh(T)) - 2\log \left( \cosh \left( \frac{\kappa T}{2} \right) \right)
\]

Remarks on the QL-plot (a) with the logit of the cumulative function on y-axis:

- Logit-scale: \( \logit(0.001, 0.01, 0.05, 0.95, 0.99, 0.999) = (-6.9, -4.6, -2.9, 0, 2.9, 4.6, 6.0) \)
- The logit of the logistic function \( \left( \frac{X}{2} \right) \) is the dashed bisecting line
- The Laplace-Gauss function (dashed lines+points) is concave and then convex
- The power hyperbola logistic functions are convex and then concave (\( \Rightarrow \) subexponential functions)
- The curvatures of the power hyperbola logistic functions depend on parameter \( \kappa \) and are well separated
Comparison with Cauchy distribution \((\kappa = 1)\)

**Figure:** K1 model \((\kappa = 1)\) and Cauchy functions: (a) Cumulative functions - (b) Densities

\[
F(x) = \frac{1}{1 - \frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1}} \quad x = \frac{2p-1}{p(1-p)} \quad f(x) = \frac{1}{x^2 + 4 + 2\sqrt{x^2 + 4}} \quad f(x) = \frac{1}{\left(\frac{1}{p}\right)^2 + \left(\frac{1}{1-p}\right)^2}
\]

\[
G(x) = 0, 5 + \frac{1}{\pi} \arctan\left(\frac{x}{\pi}\right) \quad x = \pi \tan(\pi (p - 0.5)) \quad g(x) = \frac{1}{\pi^2 + x^2} \quad g(x) = \frac{\cos^2(\pi (p - 0.5))}{\pi^2}
\]

\[
\lim_{X \to +\infty} F(X) = 1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^4} + o\left(\frac{1}{x^5}\right) \quad \lim_{X \to +\infty} G(X) = 1 - \frac{1}{x} + \frac{\pi^2}{3x^3} - \frac{\pi^4}{5x^5} + o\left(\frac{1}{x^7}\right)
\]

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Comparison with Laplace-Gauss distribution (any $\kappa$, $\gamma = 0.313$)

Figure: K1 model ($\gamma = 0, 313, \kappa = 3.2$) and Laplace-Gauss distributions ($\sigma = 1$): (a) Cumulative functions - (b) Densities

K1 and Laplace-Gauss distributions can be compared when they have same density peaks. The equality in $X = \mu$ gives $h = \frac{\zeta \sigma}{\zeta \gamma} = \frac{8 \gamma}{\sqrt{2\pi} \sigma} = 1$, or:

$$\gamma = \frac{\sqrt{2\pi}}{8} \sigma \approx 0,313 \sigma \quad \text{et} \quad \sigma = \frac{8}{\sqrt{2\pi}} \gamma \approx 3,192 \gamma$$
Comparison with centered Student distribution

Centered Student distribution with \( \nu \) degrees of freedom:

\[
f(x) = \frac{1}{\sqrt{\nu \pi}} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}
\]

\[\begin{array}{cccccc}
\nu & 1 & 2 & 3 & 5 & 7 \\
\logit(q.999) & = 6.9 & & & & \\
\logit(q.99) & = 4.6 & & & & \\
\logit(q.95) & = 2.9 & & & & \\
\logit(q.50) & = 0 & & & & \\
\logit(q.05) & = -2.9 & & & & \\
\logit(q.01) & = -4.6 & & & & \\
\logit(q.001) & = -6.9 & & & & \\
\end{array}\]

\[\begin{array}{cccccc}
\nu & 1 & 2 & 3 & 5 & 7 \\
\text{Logit(Proba)} & : & \text{Student} \ (\nu = ... ) + \text{Logistique} + \text{Gauss} \\
\end{array}\]

\[\begin{array}{cccccc}
\nu & 1 & 2 & 3 & 5 & 7 \\
\text{LogDensités} & : & \text{Student} \ (\nu = ... ) + \text{Logistique} + \text{Gauss} \\
\end{array}\]

\[\begin{array}{cccccc}
\nu & 1 & 2 & 3 & 5 & 7 \\
\text{Densités} & : & \text{Student} \ (\nu = ... ) + \text{Gauss} \\
\end{array}\]

**Figure**: Student \((\nu = 1, 2, 3, 5, 7, 25)\), Logistic and Laplace-Gauss distributions:
- (a) Logit of cumulative functions
- (b) Logdensities
- (c) Densities

- \( \nu \) can take integer values only:
  - \( \nu < 1 \) is impossible
  - \( \nu = 1 \equiv \text{Cauchy distribution} \)
  - \( \nu = 2 \approx \kappa = 2. \nu = 8 \approx \text{logistic distribution} \)
  - \( \nu = +\infty \equiv \text{Laplace-Gauss} \)
- The density peaks depend on \( \nu \)
- In practice, one can use \( \nu = (2, 3, 4, 5, 6, 7, 8) \) which cover the subexponential domain
Symmetric processes ($\kappa = 2, \kappa = 3.2, \kappa = 5$ et $\kappa = 10$) x ($\epsilon = 0$)

Figure: Processes and cumulated processes: (a) $\kappa = 2$ (b) $\kappa = 3.2$ (c) $\kappa = 5$ (d) $\kappa = 10$
Gold lingot: $\kappa = 4$, $\varepsilon = 0$, multi-scale symmetry

Figure: (a) Price of the gold lingot - (b) 100x log-returns
(c) Cumulative function ($\kappa = \alpha = \omega$) - (d) Logit of the cumulative function
(e) Cumulative function in Log-Log scale with periods day, week, month
(f) Risks at 1% et 1% over a yearly period (250 days) described by Laplace-Gauss and K1 estimates
The annual risk profiles of Société Générale and Vivendi

The analysis of the returns can be conducted over fixed periods, for instance one year (about 250 days). The tail parameter $\kappa$ describes the model curvature. q.001 is the risk at 1‰.

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Explicit models for bilateral fat-tailed distributions

R/Rmetrics Workshop - 27 June 2014 16 / 32
The yearly return and risk profiles of Vivendi

**Figure:** Logit of cumulative functions of Vivendi (2002: Jean-Marie Messier)

(a) 1998 - (b) 1999 - (c) 2000 - (d) 2001 - (e) 2002 - (f) 2003 - (g) 2004 - (h) 2005 - (i) 2006
The yearly return and risk profiles of Société Générale

Figure: Logit of cumulative functions of Société Générale
(a) 2005 - (b) 2006 - (c) 2007 - (d) 2008 - (e) 2009 - (f) 2010 - (g) 2011 - (h) 2012 - (i) 2013
The yearly risk profiles

Figure: Quantiles: (a) Gold lingot - (b) CAC40 - (c) Société Générale - (d) Vivendi estimated with Laplace-Gauss (dashed lines) and K1 (full lines) models at levels 1% et 1‰ with datasets of about 250 points per year
Comparison of K1 distribution parameters

Figure: Comparison of various parameters obtained from K1 distribution (one point = one year ≈ 250 days):
(a) Median and mean - (b) κ and kurtosis - (c) κ and ratio γ/σ
Asymmetric power functions - K2

We call *power functions* the positive functions that verify the equation:

\[
\left( y^{1/\alpha} + \frac{X - \mu}{\kappa \gamma} \right) y^{1/\omega} = 1 \quad (4)
\]

where \( \kappa \) is the harmonic mean of \( \alpha \) and \( \omega \) such that
\[
\frac{1}{\kappa} = \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\omega} \right)
\]

We call *distribution K2* the cumulative function \( F \) and the density function \( f \) defined by:

\[
p = F(X) = \frac{1}{1 + y} \quad \text{et} \quad f(X) = \frac{dF(X)}{dX} \quad (5)
\]

\( X \) has a simple form in \( y \) and \( p \). Since \( y = \frac{1-p}{p} \) and \( \log(y) = -\logit(p) \), it comes:

\[
X = \mu + \gamma \kappa \left( -y^{1/\alpha} + y^{-1/\omega} \right) = \mu + \gamma \kappa \left( -\left( \frac{p}{1-p} \right)^{-1/\alpha} + \left( \frac{p}{1-p} \right)^{1/\omega} \right)
\]

Quantile of K2 distribution

\[
X(p; \mu, \gamma, \alpha, \omega) = \mu + \gamma \kappa \left( -e^{-\logit(p)/\alpha} + e^{\logit(p)/\omega} \right) \quad (6)
\]
K3 and K4 models: another form for K2 and an extension to K1

\(\alpha\) and \(\omega\) are naturally highly correlated within model K2. Consider the therm\(s\) \(\epsilon\) and \(\delta\):

\[
\frac{1}{\kappa} = \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\omega} \right) \quad \text{et} \quad \delta = \frac{\epsilon}{\kappa} = \frac{1}{2} \left( -\frac{1}{\alpha} + \frac{1}{\omega} \right)
\]

It comes:

### Conversion from \(\alpha\) and \(\omega\) to and from \(\kappa\), \(\delta\) and \(\epsilon\)

\[
\begin{align*}
\frac{1}{\alpha} &= \frac{1}{\kappa} - \delta \\
\frac{1}{\omega} &= \frac{1}{\kappa} + \delta \\
\epsilon &= \frac{\alpha - \omega}{\alpha + \omega} \\
\alpha &= \frac{\kappa}{1 - \epsilon} \\
\omega &= \frac{\kappa}{1 + \epsilon}
\end{align*}
\] (7)

\(-1 < \epsilon < 1\) is a measure of the model \textit{excentricity}. It can be expressed as a %

\(-\frac{1}{\kappa} < \delta < \frac{1}{\kappa}\) is a measure of the model \textit{distorsion}. It can be expressed in % or \(\%\).

Rewrite the model K2:

### Quantiles for models K3 and K4

\[
\begin{align*}
X(p; \mu, \gamma, \kappa, \delta) &= \mu + 2\gamma\kappa \sinh \left( \frac{\logit(p)}{\kappa} \right) e^{\delta \logit(p)} \\
X(p; \mu, \gamma, \kappa, \epsilon) &= \mu + 2\gamma\kappa \sinh \left( \frac{\logit(p)}{\kappa} \right) e^{\frac{\epsilon}{\kappa} \logit(p)}
\end{align*}
\] (8)

Remark: \(e^{\delta \logit(p)} = \left( \frac{p}{1-p} \right)^{\delta}\)
Asymmetric models (K2, K3, K4)

Figure: Models K2 and K3: 
(a) Quantiles - (b) Cumulative functions (c) Densities 
(d) Quantiles derivates - (e) Logit of the cumulative functions - (f) Logdensities
Comparison with noncentral Student distribution

Noncentral Student distribution with \( \nu \) degrees of freedom and noncentral parameter \( \mu \):

\[
F_{\nu,\mu}(x) = \begin{cases} 
\frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} (-\mu \sqrt{2})^j e^{-\mu^2/2} \frac{\Gamma(j+1)}{\Gamma(1/2)} I \left( \frac{\nu}{\nu+x^2}; \frac{\nu}{2}, \frac{j+1}{2} \right), & x \geq 0 \\
1 - \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{j!} (-\mu \sqrt{2})^j e^{-\mu^2/2} \frac{\Gamma(j+1)}{\Gamma(1/2)} I \left( \frac{\nu}{\nu+x^2}; \frac{\nu}{2}, \frac{j+1}{2} \right), & x < 0 
\end{cases}
\]

Figure: Noncentral Student distribution (\( \nu = 3 \), ncp = 0, 1, 2, 4, 5)  
(a) Densities - (b) Logdensities - (c) Logit of the cumulative functions

- The noncentral parameter \( \mu \) can take any real value
- It simultaneously modifies the distribution tails AND the mode, the median and the mean
Asymmetric processes \((\kappa = 2, 3.2, 5, 10)\times(\delta = -0.04, -0.08, -0.12)\)

**Figure:** Cumulative processes: (a) \(\kappa = 2\) (b) \(\kappa = 3.2\) (c) \(\kappa = 5\) (d) \(\kappa = 10\)
Euro-Dollar: \( \alpha = 9.1, \omega = 8.1, \epsilon = 6\% \), multi-scale asymmetry

Figure: (a) Euro-Dollar price - (b) 100\times\text{logreturns of Euro-Dollar}
(c) Cumulative function \( \alpha > \kappa > \omega \) - (d) Log of the cumulative function
(e) Cumulative function in Log-Log scale with periods day, week, month
(f) Risks at 1\% et 1\%\text{oo} over a yearly period (250 days) described by Laplace-Gauss and K2 estimates

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Explicit models for bilateral fat-tailed distributions
VIX: $\alpha = 5$, $\omega = 4$, $\epsilon = 11\%$, strong multi-scale asymmetry

Figure:  (a) VIX price - (b) 100×logreturns of VIX
(c) Cumulative function ($\alpha > \kappa > \omega$) - (d) Logit of the cumulative function
(e) Cumulative function in Log-Log scale with periods day, week, month
(f) Risks at 1% et 1‰ over a yearly period (250 days) described by Laplace-Gauss and K2 estimates
Package FatTailsR

Miscellaneous
- `ashp`, `kashp(x, k = 1)`, `dkashp_dx`, `invlogit(x)`, `logit(p)`

Power hyperbolas, power hyperbolic functions and their inverses
- `exphp(x, k = 1)`, `coshp`, `sinhp`, `tanhp`, `sechp`, `cosechp`, `cotanhp`
- `loghp(x, k = 1)`, `acoshp`, `asinhp`, `atanhp`, `asechp`, `acosechp`, `acotanhp`

Logishp and Kiener1 symmetric functions, without or with parameter m and g
- `d,p,q,r logishp( xqpn, k = 1)`
- `d,p,q,r kiener1( xqpn, m = 0, g = 1, k = 3.2)`

Kiener2, Kiener3, Kiener4 asymmetric functions
- `q,r kiener2( pn, m = 0, g = 1, a = 3.2, w = 3.2)`
- `q,r kiener3( pn, m = 0, g = 1, k = 3.2, d = 0)`
- `q,r kiener4( pn, m = 0, g = 1, k = 3.2, e = 0)`

Parameter estimation
- `laplacegaussnorm(X)`
- `regkienerLX( X, model = "k4", dgts = c(3, 3, 1, 1, 1, 3, 2, 4, 4, 2, 2), maxk = 10, mink = 0.7, app = 0)`

⇒ Available on CRAN at:
http://cran.r-project.org/web/packages/FatTailsR/index.html
Package FatTailsR: \textit{regkienerLX} and SMI indice

\begin{verbatim}
> library(FatTailsR)
> price2returns <- function(x) { 100*diff(log(x)) }
> j <- 2  # 1=DAX, 2=SMI, 3=CAC, 4=FTSE
> X <- price2returns(EuStockMarkets[,2])
> reg <- regkienerLX( X, model = "k2" )
> attributes(reg)

$names
[1]   "dfrXP"   "dfrXL"   "dfrXR"   "dfrEP"   "dfrEL"   "dfrED"   "regk0"   "coefk0"   "vcovk0"
[10]  "vcovk0m" "m cork0" "coefk"   "coefk1"   "coefk2"   "coefk3"   "coefk4"   "quantk"   "coefr"
[19]  "coefr1"   "coefr2"   "coefr3"   "coefr4"   "quantr"   "dfrQkPk"   "dfrQkLk"

$class
[1] "clregk"

> reg$coefr2
   m   g   a   w
0.089 0.223 3.800 4.400

> reg$coefr
   m   g   a   k   w   d   e
0.089 0.223 3.800 4.100 4.400 -0.018 -0.070

> reg$quantr
     q.0001 q.0005 q.001 q.005 q.01 q.05 q.50 q.95 q.99 q.995 q.999 q.9995 q.9999
-10.00  -6.44  -5.30  -3.29  -2.63  -1.42  0.09  1.44  2.39  2.88  4.29  5.06  7.33
\end{verbatim}
Conclusion

We have presented:

- Power hyperbolas, power hyperbolic functions and fat-tailed distributions which use the **median** as pivotal value
  - A symmetric model whose all representations have explicit forms (pdf, cdf, quantiles, ...)
  - A few asymmetric models whose quantile functions have explicit forms
- Very accurate models for distributions of returns. Accurate estimates of risks
- A new R package: **FatTailsR**

Some open questions:

- **Characteristic functions**, moments, Bayesian prior and posterior
- Multivariate models, copules. But which **scalar product** and which **correlation**?
  
  Correlation of central parameter $\gamma$, tail parameter $\kappa$ or quantile value $q_{1\%}$?

- **Stochastic processes.** Here, $\mu \gamma \kappa \epsilon$ are not $m \sigma$ !!
  
  What name for these new processes: $\mu \gamma \kappa \epsilon$-Garch or maybe Garck?

  \[
  \frac{dS_t}{S_t} = \mu_m dt + \gamma dW_t \quad \text{with} \quad dW_t \approx \approx 2\kappa \sinh \left( \frac{\text{logit}(\rho_t)}{\kappa} \right) \ e^{\frac{\epsilon}{\kappa} \text{logit}(\rho_t)}
  \]

- Could the nonlinear variation of parameters ($\gamma/\sigma$, $\kappa$) presented at slide 20 explain the **volatility smile** of derivatives products?

Potential applications:

- Market risk (Bâle III, bcbs240, Jan-Fev. 2013), portfolio management, derivatives, ...
- Add new features to the package
Package **FatTailsR**

- Version 1.0-3 (14 July 2014) available on CRAN at:
  - http://cran.r-project.org/web/packages/FatTailsR/index.html

To cite and download this document:

  http://www.inmodelia.com/fattailsr-en.html
Thank you for your attention!

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